Lecture 29

Uniqueness Theorem

The uniqueness of a solution to a linear system of equations is an important concept in mathematics. Under certain conditions, ordinary differential equation partial differential equation and matrix equations will have unique solutions. But uniqueness is not always guaranteed as we shall see. This issue is discussed in many math books and linear algebra books [70,82]. The proof of uniqueness for Laplace and Poisson equations are given in [30,49] which is slightly different from electrodynamic problems.

To quote Star Trek, we need to know who the real McCoy is![1]

29.1 The Difference Solutions to Source-Free Maxwell's Equations

In this section, we will prove uniqueness theorem for electrodynamic problems [31,34,45,60,76]. First, let us assume that there exist two solutions in the presence of one set of common impressed sources $\mathbf{J}_i$ and $\mathbf{M}_i$. Namely, these two solutions are $\mathbf{E}^a$, $\mathbf{H}^a$, $\mathbf{E}^b$, $\mathbf{H}^b$. Both of them satisfy Maxwell's equations and the same boundary conditions. Are $\mathbf{E}^a = \mathbf{E}^b$, $\mathbf{H}^a = \mathbf{H}^b$?

To study the uniqueness theorem, we consider general linear anisotropic inhomogeneous media, where the tensors $\overline{\boldsymbol{\mu}}$ and $\overline{\boldsymbol{\varepsilon}}$ can be complex so that lossy media can be included. In the frequency domain, it follows that

$$\nabla \times \mathbf{E}^a = -j\omega\overline{\boldsymbol{\mu}} \cdot \mathbf{H}^a - \mathbf{M}_i \tag{29.1.1}$$

$$\nabla \times \mathbf{E}^b = -j\omega\overline{\boldsymbol{\mu}} \cdot \mathbf{H}^b - \mathbf{M}_i \tag{29.1.2}$$

$$\nabla \times \mathbf{H}^a = j\omega\overline{\boldsymbol{\varepsilon}} \cdot \mathbf{E}^a + \mathbf{J}_i \tag{29.1.3}$$

$$\nabla \times \mathbf{H}^b = j\omega\overline{\boldsymbol{\varepsilon}} \cdot \mathbf{E}^b + \mathbf{J}_i \tag{29.1.4}$$

[1]This phrase was made popular to the baby-boom generation, or the Trekkies by Star Trek. It actually refers to an African American inventor.

By taking the difference of these two solutions, we have

$$\nabla \times (\mathbf{E}^a - \mathbf{E}^b) = -j\omega\overline{\boldsymbol{\mu}} \cdot (\mathbf{H}^a - \mathbf{H}^b) \tag{29.1.5}$$

$$\nabla \times (\mathbf{H}^a - \mathbf{H}^b) = j\omega\overline{\boldsymbol{\varepsilon}} \cdot (\mathbf{E}^a - \mathbf{E}^b) \tag{29.1.6}$$

Or alternatively, defining $\delta\mathbf{E} = \mathbf{E}^a - \mathbf{E}^b$ and $\delta\mathbf{H} = \mathbf{H}^a - \mathbf{H}^b$, we have

$$\nabla \times \delta\mathbf{E} = -j\omega\overline{\boldsymbol{\mu}} \cdot \delta\mathbf{H} \tag{29.1.7}$$

$$\nabla \times \delta\mathbf{H} = j\omega\overline{\boldsymbol{\varepsilon}} \cdot \delta\mathbf{E} \tag{29.1.8}$$

The difference solutions, $\delta\mathbf{E}$ and $\delta\mathbf{H}$, satisfy the original source-free Maxwell's equations.

To prove uniqueness, we would like to find a simplifying expression for $\nabla \cdot (\delta\mathbf{E} \times \delta\mathbf{H}^*)$. By using the product rule for divergence operator, it can be shown that

$$\nabla \cdot (\delta\mathbf{E} \times \delta\mathbf{H}^*) = \delta\mathbf{H}^* \cdot \nabla \times \delta\mathbf{E} - \delta\mathbf{E} \cdot \nabla \times \delta\mathbf{H}^* \tag{29.1.9}$$

Then by taking the left dot product of $\delta\mathbf{H}^*$ with (29.1.7), and then the left dot product of $\delta\mathbf{E}^*$ with the complex conjugation of (29.1.8), we obtain

$$\begin{aligned} \delta\mathbf{H}^* \cdot \nabla \times \delta\mathbf{E} &= -j\omega\delta\mathbf{H}^* \cdot \overline{\boldsymbol{\mu}} \cdot \delta\mathbf{H} \\ \delta\mathbf{E} \cdot \nabla \times \delta\mathbf{H}^* &= -j\omega\delta\mathbf{E} \cdot \overline{\boldsymbol{\varepsilon}}^* \cdot \delta\mathbf{E}^* \end{aligned} \tag{29.1.10}$$

Now, taking the difference of the above, we get

$$\begin{aligned} \delta\mathbf{H}^* \cdot \nabla \times \delta\mathbf{E} - \delta\mathbf{E} \cdot \nabla \times \delta\mathbf{H}^* &= \nabla \cdot (\delta\mathbf{E} \times \delta\mathbf{H}^*) \\ &= -j\omega\delta\mathbf{H}^* \cdot \overline{\boldsymbol{\mu}} \cdot \delta\mathbf{H} + j\omega\delta\mathbf{E} \cdot \overline{\boldsymbol{\varepsilon}}^* \cdot \delta\mathbf{E}^* \end{aligned} \tag{29.1.11}$$

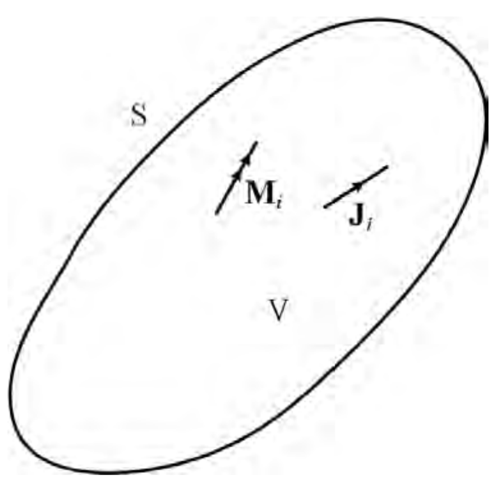

Figure 29.1: Geometry for proving the uniqueness theorem. We like to know the boundary conditions needed on S in order to guarantee the uniqueness of the solution in V.

Next, integrating the above equation over a volume V bounded by a surface S as shown in Figure 29.1. Two scenarios are possible: one that the volume V contains the impressed sources, and two, that the sources are outside the volume V. After making use of Gauss' divergence theorem, we arrive at

$$\begin{aligned}\iint_V \nabla\cdot(\delta\mathbf{E}\times\delta\mathbf{H}^*)dV &= \oiint_S(\delta\mathbf{E}\times\delta\mathbf{H}^*)\cdot d\mathbf{S}\\ &= \iiint_V[-j\omega\delta\mathbf{H}^*\cdot\overline{\boldsymbol{\mu}}\cdot\delta\mathbf{H}+j\omega\delta\mathbf{E}\cdot\overline{\boldsymbol{\varepsilon}}^*\cdot\delta\mathbf{E}^*]dV \end{aligned} \tag{29.1.12}$$

And next, we would like to know the kind of boundary conditions that would make the left-hand side equal to zero. It is seen that the surface integral on the left-hand side will be zero if:[2]

1. If $\hat{n}\times\mathbf{E}$ is specified over S for the two possible solutions, so that $\hat{n}\times\mathbf{E}_a=\hat{n}\times\mathbf{E}_b$. Then $\hat{n}\times\delta\mathbf{E}=0$, which is the PEC boundary condition for $\delta\mathbf{E}$, and then[3]
$\oiint_S(\delta\mathbf{E}\times\delta\mathbf{H}^*)\cdot\hat{n}dS=\oiint_S(\hat{n}\times\delta\mathbf{E})\cdot\delta\mathbf{H}^*dS=0.$

2. If $\hat{n}\times\mathbf{H}$ is specified over S for the two possible solutions, so that $\hat{n}\times\mathbf{H}_a=\hat{n}\times\mathbf{H}_b$. Then $\hat{n}\times\delta\mathbf{H}=0$, which is the PMC boundary condition for $\delta\mathbf{H}$, and then
$\oiint_S(\delta\mathbf{E}\times\delta\mathbf{H}^*)\cdot\hat{n}dS=-\oiint_S(\hat{n}\times\delta\mathbf{H}^*)\cdot\delta\mathbf{E}dS=0.$

3. Let the surface S be divided into two surfaces S_1 and S_2.[4] If $\hat{n}\times\mathbf{E}$ is specified over S_1, and $\hat{n}\times\mathbf{H}$ is specified over S_2. Then $\hat{n}\times\delta\mathbf{E}=0$ (PEC boundary condition) on S_1, and $\hat{n}\times\delta\mathbf{H}=0$ (PMC boundary condition) on S_2. Therefore, the left-hand side becomes
$\oiint_S(\delta\mathbf{E}\times\delta\mathbf{H}^*)\cdot\hat{n}dS=\iint_{S_1}+\iint_{S_2}=\iint_{S_1}(\hat{n}\times\delta\mathbf{E})\cdot\delta\mathbf{H}^*dS$
$-\iint_{S_2}(\hat{n}\times\delta\mathbf{H}^*)\cdot\delta\mathbf{E}dS=0.$

Thus, under the above three scenarios, the left-hand side of (29.1.12) is zero, and then the right-hand side of (29.1.12) becomes

$$\iiint_V[-j\omega\delta\mathbf{H}^*\cdot\overline{\boldsymbol{\mu}}\cdot\delta\mathbf{H}+j\omega\delta\mathbf{E}\cdot\overline{\boldsymbol{\varepsilon}}^*\cdot\delta\mathbf{E}^*]dV=0 \tag{29.1.13}$$

For lossless media, $\overline{\boldsymbol{\mu}}$ and $\overline{\boldsymbol{\varepsilon}}$ are hermitian tensors (or matrices[5]), then it can be seen, using the properties of hermitian matrices or tensors, that $\delta\mathbf{H}^*\cdot\overline{\boldsymbol{\mu}}\cdot\delta\mathbf{H}$ and $\delta\mathbf{E}\cdot\overline{\boldsymbol{\varepsilon}}^*\cdot\delta\mathbf{E}^*$ are purely real. Taking the imaginary part of the above equation yields

$$\iiint_V[-\delta\mathbf{H}^*\cdot\overline{\boldsymbol{\mu}}\cdot\delta\mathbf{H}+\delta\mathbf{E}\cdot\overline{\boldsymbol{\varepsilon}}^*\cdot\delta\mathbf{E}^*]dV=0 \tag{29.1.14}$$

[2]In the following, please be reminded that PEC stands for "perfect electric conductor", while PMC stands for "perfect magnetic conductor". PMC is the dual of PEC. Also, a fourth case of impedance boundary condition is possible, which is beyond the scope of this course. Interested readers may consult Chew, Theory of Microwave and Optical Waveguides [76].

[3]Using the vector identity that $\mathbf{a}\cdot(\mathbf{b}\times c)=\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})=\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})$.

[4]In math parlance, $S_1\cup S_2=S$.

[5]Tensors are a special kind of matrices.

The above two terms correspond to stored magnetic field energy and stored electric field energy in the difference solutions $\delta\mathbf{H}$ and $\delta\mathbf{E}$, respectively. The above being zero does not imply that $\delta\mathbf{H}$ and $\delta\mathbf{E}$ are zero.

For resonance solutions, the stored electric energy can balance the stored magnetic energy. The above resonance solutions are those of the difference solutions satisfying PEC or PMC boundary condition or mixture thereof. Also, they are the resonance solutions of the source-free Maxwell's equations (29.1.7). Therefore, $\delta\mathbf{H}$ and $\delta\mathbf{E}$ need not be zero, even though (29.1.14) is zero. This happens when we encounter solutions that are the resonant modes of the volume V bounded by surface S.

29.2 Conditions for Uniqueness

Uniqueness can only be guaranteed if the medium is lossy as shall be shown later. It is also guaranteed if lossy impedance boundary conditions are imposed.[6] First we begin with the isotropic case.

29.2.1 Isotropic Case

It is easier to see this for lossy isotropic media. Then (29.1.13) simplifies to

$$\iiint_V [-j\omega\mu|\delta\mathbf{H}|^2 + j\omega\varepsilon^*|\delta\mathbf{E}|^2]dV = 0 \tag{29.2.1}$$

For isotropic lossy media, $\mu = \mu' - j\mu''$ and $\varepsilon = \varepsilon' - j\varepsilon''$. Taking the real part of the above, we have from (29.2.1) that

$$\iiint_V [-\omega\mu''|\delta\mathbf{H}|^2 - \omega\varepsilon''|\delta\mathbf{E}|^2]dV = 0 \tag{29.2.2}$$

Since the integrand in the above is always negative definite, the integral can be zero only if

$$\delta\mathbf{E} = 0, \quad \delta\mathbf{H} = 0 \tag{29.2.3}$$

everywhere in V, implying that $\mathbf{E}_a = \mathbf{E}_b$, and that $\mathbf{H}_a = \mathbf{H}_b$. Hence, it is seen that uniqueness is guaranteed only if the medium is lossy. The physical reason is that when the medium is lossy, a pure time-harmonic solution cannot exist due to loss. The modes, which are the source-free solutions of Maxwell's equations, are decaying sinusoids.

Notice that the same conclusion can be drawn if we make μ'' and ε'' negative. This corresponds to active media, and uniqueness can be guaranteed for a time-harmonic solution. In this case, no time-harmonic solution exists, and the resonant solution is a growing sinusoid.

29.2.2 General Anisotropic Case

The proof for general anisotropic media is more complicated. For the lossless anisotropic media, we see that (29.1.13) is purely imaginary. However, when the medium is lossy, this same equation will have a real part. Hence, we need to find the real part of (29.1.13) for the general lossy case.

[6] See Chew, Theory of Microwave and Optical Waveguides.

About taking the Real and Imaginary Parts of a Complicated Expression

To this end, we digress on taking the real and imaginary parts of a complicated expression. Here, we need to find the complex conjugate[7] of (29.1.13), which is scalar, and add it to itself to get its real part. The complex conjugate of the scalar

$$c = \delta\mathbf{H}^* \cdot \overline{\boldsymbol{\mu}} \cdot \delta\mathbf{H} \tag{29.2.4}$$

is[8]

$$c^* = \delta\mathbf{H} \cdot \overline{\boldsymbol{\mu}}^* \cdot \delta\mathbf{H}^* = \delta\mathbf{H}^* \cdot \overline{\boldsymbol{\mu}}^\dagger \cdot \delta\mathbf{H} \tag{29.2.5}$$

Similarly, the complex conjugate of the scalar

$$d = \delta\mathbf{E} \cdot \overline{\boldsymbol{\varepsilon}}^* \cdot \delta\mathbf{E}^* = \delta\mathbf{E}^* \cdot \overline{\boldsymbol{\varepsilon}}^\dagger \cdot \delta\mathbf{E} \tag{29.2.6}$$

is

$$d^* = \delta\mathbf{E}^* \cdot \overline{\boldsymbol{\varepsilon}}^\dagger \cdot \delta\mathbf{E} \tag{29.2.7}$$

Therefore,

$$\Im m\left(\delta\mathbf{H}^* \cdot \overline{\boldsymbol{\mu}} \cdot \delta\mathbf{H}\right) = \frac{1}{2j}\delta\mathbf{H}^* \cdot \left(\overline{\boldsymbol{\mu}} - \overline{\mu}^\dagger\right) \cdot \delta\mathbf{H} \tag{29.2.8}$$

$$\Im m\left(\delta\mathbf{E} \cdot \overline{\varepsilon} \cdot \delta\mathbf{E}^*\right) = \frac{1}{2j}\delta\mathbf{E}^* \cdot \left(\overline{\varepsilon} - \overline{\varepsilon}^\dagger\right) \cdot \delta\mathbf{E} \tag{29.2.9}$$

and similarly for the real part.

Finally, after taking the complex conjugate of the scalar quantity (29.1.13) and adding it to itself, we have

$$\iiint_V [-j\omega\delta\mathbf{H}^* \cdot (\overline{\boldsymbol{\mu}} - \overline{\boldsymbol{\mu}}^\dagger) \cdot \delta\mathbf{H} - j\omega\delta\mathbf{E}^* \cdot (\overline{\varepsilon} - \overline{\varepsilon}^\dagger) \cdot \delta\mathbf{E}]dV = 0 \tag{29.2.10}$$

For lossy media, $-j(\overline{\boldsymbol{\mu}} - \overline{\boldsymbol{\mu}}^\dagger)$ and $-j(\overline{\varepsilon} - \overline{\varepsilon}^\dagger)$ are hermitian positive matrices. Hence the integrand is always positive definite, and the above equation cannot be satisfied unless $\delta\mathbf{H} = \delta\mathbf{E} = 0$ everywhere in V. Thus, uniqueness is guaranteed in a lossy anisotropic medium.

Similar statement can be made as the isotropic case if the medium is active. Then the integrand is positive definite, and the above equation cannot be satisfied unless $\delta\mathbf{H} = \delta\mathbf{E} = 0$ everywhere in V and hence, uniqueness is satisfied.

29.3 Hind Sight

The proof of uniqueness for Maxwell's equations is very similar to the proof of uniqueness for a matrix equation [70]

$$\overline{\mathbf{A}} \cdot \mathbf{x} = \mathbf{b} \tag{29.3.1}$$

[7] Also called hermitian conjugate.

[8] To arrive at these expressions, one makes use of the matrix algebra rule that if $\overline{\mathbf{D}} = \overline{\mathbf{A}} \cdot \overline{\mathbf{B}} \cdot \overline{\mathbf{C}}$, then $\overline{\mathbf{D}}^t = \overline{\mathbf{C}}^t \cdot \overline{\mathbf{B}}^t \cdot \overline{\mathbf{A}}^t$. This is true even for non-square matrices. But for our case here, $\overline{\mathbf{A}}$ is a 1×3 row vector, and $\overline{\mathbf{C}}$ is a 3×1 column vector, and $\overline{\mathbf{B}}$ is a 3×3 matrix. In vector algebra, the transpose of a vector is implied. Also, in our case here, $\overline{\mathbf{D}}$ is a scalar, and hence, its transpose is itself.

If a solution to a matrix equation exists without excitation, namely, when $\mathbf{b} = 0$, then the solution is the null space solution [70], namely, $\mathbf{x} = \mathbf{x}_N$. In other words,

$$\overline{\mathbf{A}} \cdot \mathbf{x}_N = 0 \tag{29.3.2}$$

These null space solutions exist without a "driving term" $\mathbf{b}$ on the right-hand side. For Maxwell's Equations, $\mathbf{b}$ corresponds to the source terms. They are like the homogeneous solution of an ordinary differential equation or a partial differential equation [82]. In an enclosed region of volume V bounded by a surface S, homogeneous solutions are the resonant solutions of this Maxwellian system. When these solutions exist, they give rise to non-uniqueness.

Also, notice that (29.1.7) and (29.1.8) are Maxwell's equations without the source terms. In a closed region V bounded by a surface S, only resonance solutions for $\delta\mathbf{E}$ and $\delta\mathbf{H}$ with the relevant boundary conditions can exist when there are no source terms.

As previoulsy mentioned, one way to ensure that these resonant solutions are eliminated is to put in loss or gain. When loss or gain is present, then the resonant solutions are decaying sinusoids or growing sinusoids. Since we are looking for solutions in the frequency domain, or time harmonic solutions, we are only looking for the solution on the real ω axis on the complex ω plane. These non-sinusoidal solutions are outside the solution space: They are not part of the time-harmonic solutions we are looking for. Therefore, there are no resonant null-space solutions.

We see that the source of non-uniqueness is the homogeneous solutions or the resonance solutions of the system. These solutions are non-causal, and they are there in the system since the beginning of time to time ad infinitum. One way to remove these resonance solutions is to set them to zero at the beginning by solving an initial value problem. However, this has to be done in the time domain. One reason for non-uniqueness is because we are seeking the solutions in the frequency domain.

29.3.1 Connection to Poles of a Linear System

The output to input of a linear system can be represented by a transfer function $H(\omega)$ [47,160]. If $H(\omega)$ has poles, and if the system is lossless, the poles are on the real axis. Therefore, when $\omega = \omega_{\text{pole}}$, the function $H(\omega)$ becomes undefined. In other words, one can add a constant term to the output, and the ratio between output to input is still infinity. This also gives rise to non-uniqueness of the output with respect to the input. Poles usually correspond to resonant solutions, and hence, the non-uniqueness of the solution is intimately related to the non-uniqueness of Maxwell's equations at the resonant frequencies of a structure. This is illustrated in the upper part of Figure 29.2.

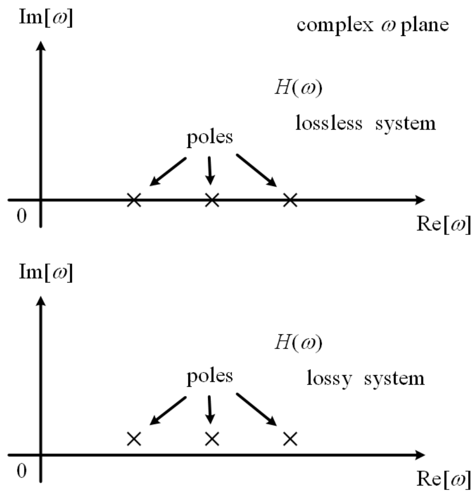

Figure 29.2: The non-uniqueness problem is intimately related to the locations of the poles of a transfer function being on the real axis.

If the input function is $f(t)$, with Fourier transform $F(\omega)$, then the output $y(t)$ is given by the following Fourier integral, viz.,

$$y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{j\omega t} H(\omega) F(\omega) \tag{29.3.3}$$

where the Fourier inversion integral path is on the real axis on the complex ω plane. The Fourier inversion integral is undefined or non-unique.

However, if loss is introduced, these poles will move away from the real axis as shown in the lower part of Figure 29.2. Then the transfer function is uniquely determined for all frequencies on the real axis. In this way, the Fourier inversion integral in (29.3.3) is well defined, and uniqueness of the solution is guaranteed.

When the poles are located on the real axis yielding possibly non-unique solutions, a remedy to this problem is to use Laplace transform technique [47]. The Laplace transform technique allows the specification of initial values, which is similar to solving the problem as an initial value problem (IVP).

29.4 Radiation from Antenna Sources and Radiation Condition

The above uniqueness theorem guarantees that if we have some antennas with prescribed current sources on them, the radiated field from these antennas are unique. To see how this can come about, we first study the radiation of sources into a region V bounded by a large surface S_{inf} as shown in Figure 29.3 [34].

Even when $\hat{n} \times \mathbf{E}$ or $\hat{n} \times \mathbf{H}$ are specified on the surface at S_{inf}, the solution is nonunique because the volume V bounded by S_{inf}, can have many resonant solutions. In fact, the region will be replete with resonant solutions as one makes S_{inf} become very large. The way to remove these resonant solutions is to introduce an infinitesimal amount of loss in region V. Then these resonant solutions will disappear from the real ω axis, where we seek a time-harmonic solution. Now we can take S_{inf} to infinity, and the solution will always be unique even if the loss is infinitesimally small.

Notice that if $S_{\text{inf}} \to \infty$, the waves that leave the sources will never be reflected back because of the small amount of loss. The radiated field will just disappear into infinity. This is just what radiation loss is: power that propagates to infinity, but never to return. In fact, one way of guaranteeing the uniqueness of the solution in region V when S_{inf} is infinitely large, or that V is infinitely large is to impose the radiation condition: the waves that radiate to infinity are outgoing waves only, and never do they return. This is also called the Sommerfeld radiation condition [161]. Uniqueness of the field outside the sources is always guaranteed if we assume that the field radiates to infinity and never to return. This is equivalent to solving the cavity solutions with an infinitesimal loss, and then letting the size of the cavity become infinitely large.

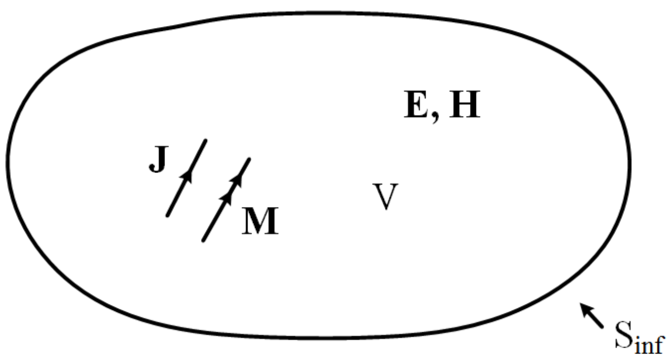

Figure 29.3: The solution for antenna radiation is unique because we impose the Sommerfeld radiation condition when seeking the solution. This is equivalent to assuming an infinitesimal loss when seeking the solution in V.